Exercise 1.1.7. Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Denote $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$ by A. For any matrix $B = (v_1, v_2, v_3)$, if we multiply A on the right, we get a matrix $(v_1, v_1 + v_2, v_1 + v_2 + v_3)$. Thus, the answer is obviously

$$\begin{bmatrix} 1 & n & (n+1)n/2 \\ & 1 & n \\ & & 1 \end{bmatrix}.$$

Proof. $A^n = \begin{bmatrix} 1 & n & (n+1)n/2 \\ & 1 & n \\ & & 1 \end{bmatrix}$. We prove this by induction.

1. **Base Case**: This is obviously true when $n = 1$.
2. **Induction Step**: Assume this holds for $n = k$. For $n = k + 1$:

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{bmatrix} 1 & k & (k+1)k/2 \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & k+1 & (k+2)(k+1)/2 \\ & 1 & k+1 \\ & & 1 \end{bmatrix}. \end{aligned}$$

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Exercise 1.1.16. A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible.

Proof. Note that

$$\begin{aligned} I &= I^{2k-1} + A^{2k-1} \\ &= (I + A)(A^{2k-2} - A^{2k-3} + \ldots + I) \\ &= (A^{2k-2} - A^{2k-3} + \ldots + I)(I + A), \end{aligned}$$

which implies $A^{2k-2} - A^{2k-3} + \ldots + I$ is the inverse of $I + A$.

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Exercise 1.1.17. (a) Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

(b) Prove that there is no matrix C such that $AC = I_3$.

(a) is easy to obtain since the nullspace of A^{T} is not 0.
We now prove (b).

Proof. Suppose that $AC = I_3$. Note that $\operatorname{rank}(AC) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(C)\} \leq 2 < \operatorname{rank}(I_3) = 3$, which leads to a contradiction. ■