Exercise 2.1.5. Assume that the equation $xyz = 1$ holds in a group G. Does it follows that $yzx = 1$? That $yxz = 1$?

Solution. $yzx = 1$ is true since yz has the only inverse $(yz)^{-1} = x$ in G.

$yzx = 1$ may not be true. Otherwise, it implies

$$\begin{aligned} yxz &= 1 = xyz \\ \Leftrightarrow yx &= xy, \end{aligned}$$

which is not true since we can arbitrarily choose x, y and G is not necessarily abelian.

Exercise 2.1.7. Let S be any set. Prove that the law of composition defined by $ab = a$ is associative.

Proof. For all $a, b, c \in S$, we have:

$$\begin{aligned} (ab)c &= ac = a \\ a(bc) &= ab = a. \end{aligned}$$

□

Exercise 2.2.1. Determine the elements of the cyclic group generated by the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ explicitly.

Solution. Denote $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ by A. We have:

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \\ A^3 &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, \\ A^4 &= A^3 A = (-I)A = -A, \\ A^5 &= -A^2, \\ A^6 &= (A^3)^2 = I. \end{aligned}$$

Thus, $\langle A \rangle = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}.$

Exercise 2.2.15. (a) In the definition of subgroup, the identity element in H is required to be the identity of G. One might require only that H have an identity element, not that it is the same as the identity in G. Show that if H has an identity at all, then it is the identity in G, so this definition would be equivalent to the one given.

(b) Show the analogous thing for inverses.

Proof. (a) Assume that there exists an identity $e' \in H$. We have

$$e'e' = e' = e'1,$$

then we know $e' = 1$ by the cancellation law.

(b) For any element $a \in H$, there exists another $b \in H$ such that $ab = ba = 1$. We also know $aa^{-1} = a^{-1}a = 1$, thus we conclude $b = a^{-1}$ by the cancellation law.

□

Exercise 2.2.20 (a). Let a, b be elements of an abelian group of orders m, n respectively. What can you say about the order of their product ab?

I am not sure about the answer to this question. It is easy to see that $(ab)^{\frac{mn}{(m,n)}} = a^{m\frac{n}{(m,n)}} b^{n\frac{m}{(m,n)}} = 1$ and thus the order of ab divides $\frac{mn}{(m,n)}$, where (m, n) represents the greatest common divisor of m and n. But I still don't know what the order exactly is.