Exercise 2.3.1. Prove that the additive group $\mathbb{R}^+$ of real numbers is isomorphic to the multiplicative group P of positive numbers.

Proof. Let $\varphi : \mathbb{R} \to P$ be the exponential function:

$$\varphi(x) = e^x,$$

where $x \in \mathbb{R}$. We now show this function is an isomorphism:

1. φ is bijective since $\varphi^{-1}(y) = \ln y, y > 0$.
2. $\varphi(x + y) = e^{x+y} = e^x e^y = \varphi(x)\varphi(y)$.

□

Exercise 2.3.11. Prove that the set of Aut G of automorphisms of a group G forms a group, the law of composition being composition of functions.

Proof. We prove this by showing that the following properties are satisfied:

1. Closure: For any two functions $f, g \in \text{Aut } G$, it is obvious that $f \circ g$ is bijective since f and g are bijective. For any two elements $x, y \in G$, we have

$$\begin{aligned} f \circ g(x \cdot y) &= f(g(x \cdot y)) \\ &= f(g(x) \cdot g(y)) \\ &= f(g(x)) \cdot f(g(y)) \\ &= f \circ g(x) \cdot f \circ g(y). \end{aligned}$$

2. Associative law: This is satisfied since the composition of functions is associative.
3. Identity: Let $i : G \to G$ be $i(x) = x, x \in G$.
4. Inverses: For all $f \in \text{Aut } G$, we claim that the inverse of f is f^{-1}. To show this, we say for any $x, y \in G$:

$$\begin{aligned} f^{-1}(x \cdot y) &= f^{-1}(f(f^{-1}(x)) \cdot f(f^{-1}(y))) \\ &= f^{-1}(f(f^{-1}(x) \cdot f^{-1}(y))) \\ &= f^{-1}(x) \cdot f^{-1}(y). \end{aligned}$$

□

Exercise 2.3.12. Let G be a group, and let $\varphi : G \to G$ be the map $\varphi(x) = x^{-1}$.

(a) Prove that φ is bijective.

(b) Prove that φ is an automorphism if and only if G is abelian.

proof of (a). Note that $\varphi\varphi = 1$, thus $\varphi^{-1} = \varphi$ and φ is bijective.

□

proof of (b).

$$\begin{aligned} &\varphi \text{ is automorphism} \\ \Leftrightarrow &\text{For any } x^{-1}, y^{-1} \in G, \text{ we have } \varphi(x^{-1} \cdot y^{-1}) = (x^{-1} \cdot y^{-1})^{-1} = y \cdot x \\ = &\varphi(x^{-1}) \cdot \varphi(y^{-1}) = x \cdot y \\ \Leftrightarrow &x \cdot y = y \cdot x. \end{aligned}$$

□

Exercise 2.4.3. Prove that the kernel and image of a homomorphism are subgroups.

Proof. Assume there is a homomorphism $f : G \to G'$. We first show that $\ker f$ is a subgroup of G.

1. Closure: If $a, b \in \ker f$, $f(a) = f(b) = 1'$. Therefore, $f(ab) = f(a) \cdot f(b) = 1'$, showing $ab \in \ker f$.

2. Identity: For any $a \in G$, we have $f(a) = f(a \cdot 1) = f(a) \cdot f(1)$. Thus, $f(1) = 1'$ and this means $1 \in \ker f$.

3. Inverses: If $a \in G$ satisfying $f(a) = 1$, then $1' = f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1}) = f(a^{-1})$. Thus, $a^{-1} \in \ker f$.

We can see that $1' \in \text{im } f$ from above since $f(1) = 1'$. For any $a', b' \in \text{im } f$, there exist $a, b \in G$ such that $f(a) = a', f(b) = b'$. Therefore, $a' \cdot b' = f(a) \cdot f(b) = f(a \cdot b) = (a \cdot b)'$, which implies the composition is closed. Similary, we know the inverse of $a' \in \text{im } f$ since $1' = f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1}) = a' \cdot (a^{-1})'$.

□

Exercise 2.4.6. Let $f : \mathbb{R}^+ \to \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

Proof. For any 2 elements $x, y \in \mathbb{R}$, we have

$$f(x + y) = e^{i(x+y)} = e^{ix}e^{iy} = f(x)f(y).$$

$\ker f = \{2k\pi : k \in \mathbb{Z}\}$; $\text{im } f = \{e^{ix} : x \in \mathbb{R}\} = \{x : x \in \mathbb{C} \wedge |x| = 1\}$.

□

Exercise 2.4.11. Let G, H be cyclic groups, generated by elements x, y. Determine the condition on the orders m, n of x and y so that the map sending $x^i \mapsto y^i$ is a group homomorphism.

Proof. Denote this map by f. Note that $f(x^m) = f(1) = 1 = y^m$, which means that $n \mid m$. And this is also sufficient, since $f(x^{kn+i}) = y^i = f(x^i)$ for all $k \in \mathbb{Z}$ and $0 \leq i < n$.

□