Problem 1. Let V denote the Klein 4-group. Show that $\mathrm{Aut}(V)$ is isomorphic to S_3.

Proof. Consider a function $i : S_3 \to \mathrm{Aut}(V)$, which sends $\alpha \in S_3$ to f_α defined as follows:

$$f_\alpha(e) = e, f_\alpha(\tau) = g(\alpha(1)), f_\alpha(\tau') = g(\alpha(2)), f_\alpha(\tau'') = g(\alpha(3)),$$

where $g(1) = \tau, g(2) = \tau', g(3) = \tau''$. We can see f_α is an automorphism of V and $\{f_\alpha : \alpha \in S_3\} = \mathrm{Aut}(V)$.

□

Problem 2. Define $f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ by $f(A) = {}^tA^{-1}$ (where ${}^tA^{-1}$ is the transpose of A). Show that f is an automorphism, but not an inner automorphism for $n \geq 1$.

Proof. f is a bijection since $f^{-1} = f$. Moreover, we have

$$f(AB) =^t (AB)^{-1} =^t \left(B^{-1}A^{-1}\right) = \left({}^tA^{-1}\right)\left({}^tB^{-1}\right) = f(A)f(B).$$

Therefore, f is an automorphism. If f is also an inner automorphism, then there exists some $B \in \mathrm{GL}_n(\mathbb{R})$ such that $f(A) = BAB^{-1}$. However, $\det\left(BAB^{-1}\right) = \det(A) \neq \det({}^tA^{-1}) = \frac{1}{\det(A)}$ for $\det(A) \neq 1$.

□

Exercise 2.4.5. Let G be an abelian group. Prove that the n-th power map $\varphi : G \to G$ defined by $\varphi(x) = x^n$ is a homomorphism from G to itself.

Proof.

$$\varphi(x \cdot y) = (x \cdot y)^n = x^n \cdot y^n = \varphi(x) \cdot \varphi(y).$$

□