Exercise 2.5.1. Prove that the nonempty fibres of a map form a partition of the domain.

Proof. Denote by f, D the map and the domain, respectively. For all $a \in D$, $a \in f^{-1}(f(a))$, which means that the union of fibres is D. For two fibres $\bar{a} := f^{-1}(f(a))$ and $\bar{b} := f^{-1}(f(b))$, if they have a common element, then they are equal. Assume $c \in \bar{a} \cap \bar{b}$, then $f(a) = f(c) = f(b)$, showing that $\bar{a} = \bar{b}$.

□

Exercise 2.5.6. (a) Prove that the relation x conjugate to y in a group G is an equivalence relation on G.

(b) Describe the elements a whose conjugacy class (=equivalence class) consists of the element a alone.

Proof of (a).

1. $x \sim x$: $exe^{-1} = x$.
2. $x \sim y \implies y \sim x$: $gxg^{-1} = y \implies y = g^{-1}xg$, for some $g \in G$.
3. $x \sim y \wedge y \sim z \implies x \sim z$: $gxg^{-1} = y \wedge hyh^{-1} = z$, implies $(hg)\,x\,(hg)^{-1} = z$, for some $g, h \in G$.

□

Proof of (b). If $\bar{a} = \{a\}$, then for all $g \in G$, we have

$$gag^{-1} = a,$$

which shows $ga = ag$, thus $a \in \mathrm{Z}(G)$.

□

Exercise 2.6.2. Prove directly that distinct cosets do not overlap.

Proof. We show this is true for left cosets and the case of right cosets follows the same idea.

Given a subgroup $H \subset G$, if $d \in aH$ for some $d, a \in G$, then $dH \subset aH$. There exists some h such that $d = ah$, which implies that $a = dh^{-1}$, showing $aH \subset dH$. Hence, $aH = dH$. If $d \in aH \cap bH$, $aH = dH = bH$.

□

Exercise 2.6.4. Give an example showing that the left cosets and right cosets of $\mathrm{GL}_2(\mathbb{R})$ in $\mathrm{GL}_2(\mathbb{C})$ are not always equal.

Proof. Consider $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, $B = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Note that

$$\begin{aligned} BAB^{-1} &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} & \mathrm{i} \\ 1 & \end{pmatrix} \begin{pmatrix} -\mathrm{i} & \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} & \mathrm{i} \\ -\mathrm{i} & \end{pmatrix} \notin \mathrm{GL}_2(\mathbb{R}). \end{aligned}$$

□

Exercise 2.6.5. Let H, K be subgroups of a group G of orders $3, 5$ respectively. Prove that $H \cap K = \{1\}$.

Proof. Note that $H \cap K$ is also a subgroup of G. Moreover, it is a subgroup of H and K. Hence, $|H \cap K|$ divides $\gcd(3, 5) = 1$, implying $H \cap K = \{1\}$.

□